# A generalisation of the Stormer problem



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# A generalisation of the Störmer problem

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**Abstract.** Störmer's theory of charged particles in a magnetic dipole field is generalised in order to investigate charged particle trapping in the electromagnetic field of the parallel rotator. This problem amounts to the discussion of 'allowed' and 'forbidden' regions for an electrically charged particle within an axially symmetric electromagnetic field configuration described by the superposition of a magnetic dipole and an electric quadrupole field. The trapping regions are calculated using the conservation laws of energy and generalised angular momentum.

## 1. Introduction

The trapping of charged particles in the field of a homogeneously magnetised sphere which is at rest in the inertial frame of reference was first studied by Störmer [1]. His discovery of 'allowed' and 'forbidden' regions for charged particles and of regions appropriate for particle storage has become very important for interpreting the behaviour of charged particles in the neighbourhood of magnetised stellar bodies. The appearance of trapping regions in particular has become the basis for the understanding of particle inclusion inside the radiation belts of the Earth and of other planets [2].

The latter problem has been modified by Stern [3] in order to include higher moments of the magnetic field, still under the assumption of an axial symmetry. Shalimov and Shvachunov [4] have considered a different modification by superposition of a homogeneous, constant external magnetic field, again under the assumption of axial symmetry. In this context, one can also recall the work of Artemyev [5] who, in addition to the magnetic force originating in the magnetic dipole field, considered the influence of a gravitational field of a mass point for application to the dynamics of electrically charged dust particles.

We generalise Störmer's results for the electromagnetic field of a homogeneously magnetised sphere rotating with a given angular velocity $\boldsymbol{\omega}$ which is parallel to its magnetic dipole moment $\boldsymbol{\mu}$, and obtain a variety of new features. One must, however, be careful when trying to apply these features to the situation in the neighbourhood of a magnetised star since they have been found under the premise of vacuum electrodynamics whereas the plasma present in this region will, in general, have a great influence on the electromagnetic field and on the motion of charged particles. Nevertheless, the present analysis is believed to be useful for understanding certain aspects of electromagnetic processes evolving in the vicinity of a parallel rotator.

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## 2. Condition for 'allowed' and 'forbidden' regions

We consider an electromagnetic field with azimuthal symmetry represented by the electric potential

$$A_0 = A_0(R, z)$$

superimposed on the magnetic potential

$$(A_R, A_\phi, A_z) = (0, \mu R(R^2+z^2)^{-3/2}, 0)$$

of a magnetic dipole $\boldsymbol{\mu}$. We use cylindrical coordinates $R$, $\phi$ and $z$.

Under these premises, the Langrangian

$$L = -mc^2(1-v^2/c^2)^{1/2} + (e\mu/c)R^2\dot{\phi}(R^2+z^2)^{-3/2} - eA_0$$

of a particle of rest mass $m$ and electric charge $e$ does not depend explicitly on the time coordinate $t$ and the azimuthal angle $\phi$. Therefore the $z$ component of the generalised angular momentum

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = m\gamma R^2\dot{\phi} + (e\mu/c)R^2(R^2+z^2)^{-3/2} \tag{1}$$

(which obviously is nothing but the component of the generalised momentum canonically conjugate to the angle $\phi$) as well as the total energy

$$E = m\gamma c^2 + eA_0 \tag{2}$$

are conserved quantities. We will use the following definitions:

$$\gamma = (1-v^2/c^2)^{-1/2} \qquad r = (R^2+z^2)^{1/2}$$

$$\cos\theta = R/r \qquad E_{\mathrm{R}} = mc^2.$$

Now the velocity from (2) may be used to eliminate the velocity in (1) and thus obtain a relation

$$\cos\alpha = \left(\frac{cp_\phi}{r\cos\theta} - e\mu\frac{\cos\theta}{r^2}\right)[(E-eA_0)^2 - E_{\mathrm{R}}^2]^{-1/2}$$

for the cosine of the angle $\alpha$ between the velocity vector and its projection onto the equatorial plane. Since $|\cos\alpha| \leqslant 1$ the dynamical variables attributed to a physical particle must obey the inequality

$$\left(\frac{cp_\phi}{r\cos\theta} - e\mu\frac{\cos\theta}{r^2}\right)^2 \leqslant (E-eA_0)^2 - E_{\mathrm{R}}^2. \tag{3}$$

As an example we wish to consider the electric potential

$$A_0(r, \theta) = (Q/4r^3)(3\sin^2\theta - 1)$$

describing the field of an electric quadrupole moment $Q$. The latter may be thought of as produced by the rotation of a homogeneously magnetised sphere of radius $r_0$, dielectric permeability $\varepsilon_{\mathrm{d}}$, and magnetic dipole moment $\mu$ rotating at an angular velocity $\omega$ around the direction of the dipole moment:

$$Q = -\frac{4(2\varepsilon_{\mathrm{d}}+1)}{3c(2\varepsilon_{\mathrm{d}}+3)} r_0^2\omega\mu$$

as has been discussed in great detail by Deutsch [6]. The 'allowed' regions are then defined by the inequality

$$r^6(E_R^2-E^2)\cos^2\theta+r^4c^2p_\phi^2+2r^3[\tfrac{1}{4}eQE(3\sin^2\theta-1)-cp_\phi e\mu]\cos^2\theta$$
$$+r^2e^2\mu^2\cos^4\theta+\tfrac{1}{16}e^2Q^2\cos^2\theta(3\sin^2\theta-1)^2\leq 0. \tag{4}$$

## 3. Choice of parameters

We restrict ourselves to energies $E>E_R$. Particles with these energies are not bound because of the energy conservation alone, but the conservation of momentum $p_\phi$ is decisive.

With the help of

$$\rho=\frac{(E^2-E_R^2)^{1/2}}{cp_\phi}r \tag{5a}$$

which will be used to scale particle coordinates by $R_s/R=Z_s/z=\rho/r$ and the parameters defined by

$$\varepsilon=E/(E^2-E_R^2)^{1/2} \tag{5b}$$

$$\lambda=e\mu(E^2-E_R^2)^{1/2}/c^2p_\phi^2 \tag{5c}$$

$$\kappa=eQ(E^2-E_R^2)/4c^3p_\phi^3 \tag{5d}$$

then the inequality (4) for 'allowed' regions may be written in the form

$$\rho^6\cos^2\theta-\rho^4-2\rho^3[\varepsilon\kappa(3\sin^2\theta-1)-\lambda]\cos^2\theta$$
$$-\rho^2\lambda^2\cos^4\theta-\kappa^2(3\sin^2\theta-1)^2\cos^2\theta\geq 0. \tag{6}$$

Now $E>E_R$ corresponds to $\varepsilon>1$. Also one may restrict discussion to $p_\phi>0$, i.e. $\rho>0$, since the inequality (4) is invariant against the transformation

$$\begin{pmatrix}p_\phi\\ \mu\end{pmatrix}\mapsto\begin{pmatrix}-p_\phi\\ -\mu\end{pmatrix}$$

while thereby $\rho\mapsto-\rho$ and thus

$$\begin{pmatrix}\lambda\\ \kappa\end{pmatrix}\mapsto\begin{pmatrix}-\lambda\\ -\kappa\end{pmatrix}.$$

Each of the latter two parameters then may be attributed values between $-\infty$ and $+\infty$.

It should be mentioned that equations (5) can be inverted such that for every given set of parameter values ($\rho$, $\varepsilon$, $\lambda$, $\kappa$) within the aforementioned regions of definition there can be found a magnetic dipole moment $\mu$, an electric quadrupole moment $Q$ and initial conditions $r_0$, $\gamma_0$, $v_{\phi 0}$ for a particle with charge $e$ and mass $m$.

Equation (6) shows in addition to the azimuthal symmetry a mirror symmetry with respect to the equatorial plane. Therefore we restrict our analyses to the region $R_s, Z_s>0$.

## 4. Results

It can be shown with the help of Descartes' theorem (as described, for example, in [7]) that the polynomial of the sixth order in $\rho$ defining the borderline of the 'forbidden' regions always has at least one positive root. Let $\rho_a$ be the largest positive root of this polynomial. It is then obvious that the left side of the inequality (6) becomes positive for $\rho > \rho_a$. Therefore every $\rho > \rho_a$ is 'allowed'.

Alternatively, let $\rho_f$ be the smallest positive root of the aforementioned polynomial. If we then restrict ourselves to values of $\theta$ such that $\theta \neq \pi/2$ and $\sin^2\theta \neq \frac{1}{3}$, the left side of the inequality (6) obviously becomes negative for $\rho < \rho_f$. Thus every $\rho < \rho_f$ is 'forbidden'.

Now we wish to recall the shape of the trapping regions predicted by Störmer [1] for the pure magnetic dipole field, making use of the parameters introduced here. The boundary lines between 'allowed' and 'forbidden' regions are determined through the following real and positive roots of equation (6) with $\kappa = 0$:

$$\rho_{1/2} = \frac{1}{2\cos\theta} \pm \left(\frac{1}{4\cos^2\theta} - \lambda\cos\theta\right)^{1/2}$$

$$\rho_{3/4} = -\frac{1}{2\cos\theta} \pm \left(\frac{1}{4\cos^2\theta} + \lambda\cos\theta\right)^{1/2}.$$

The corresponding configuration exhibits mirror and azimuthal symmetry, and particle trapping turns out to be possible for $0 < \lambda \leq \frac{1}{4}$.

As an illustration of this case, figure 1 shows a meridian cross section of the toroidal-shaped trapping region and the outer boundary of the 'forbidden' region for $\lambda = 0.23$.

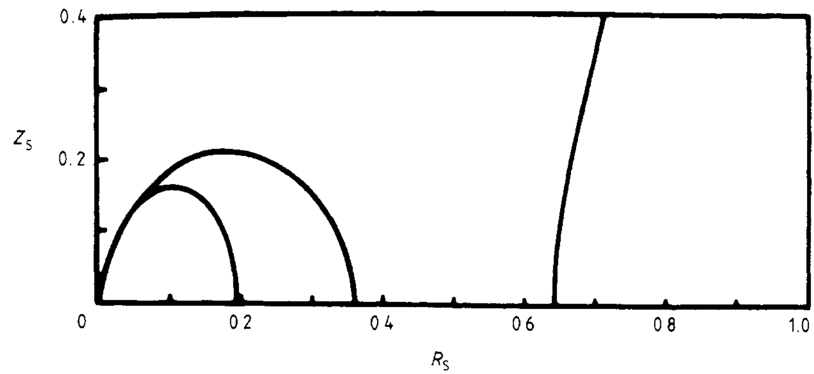


**Figure 1.** 'Allowed' and 'forbidden' regions for charged particles in a magnetic dipole field $\lambda = 0.23$.

Turning to the more general case, figure 2 shows a sequence of diagrams, for constant $\varepsilon = 2$ and $\kappa = 0.01$, as $\lambda$ increases from 0.1 to 0.215. One may imagine that this variation of $\lambda$ is produced by a corresponding growth of the magnetic dipole moment $\mu$ while all other parameters are kept constant. In $(a)$, the first of the four meridian cross sections, there is an exterior 'allowed' region on the right side of the diagram and two small tori are mirror symmetric with respect to the equatorial plane, one of which can be seen in the lower left corner. In $(b)$ these two tori have developed

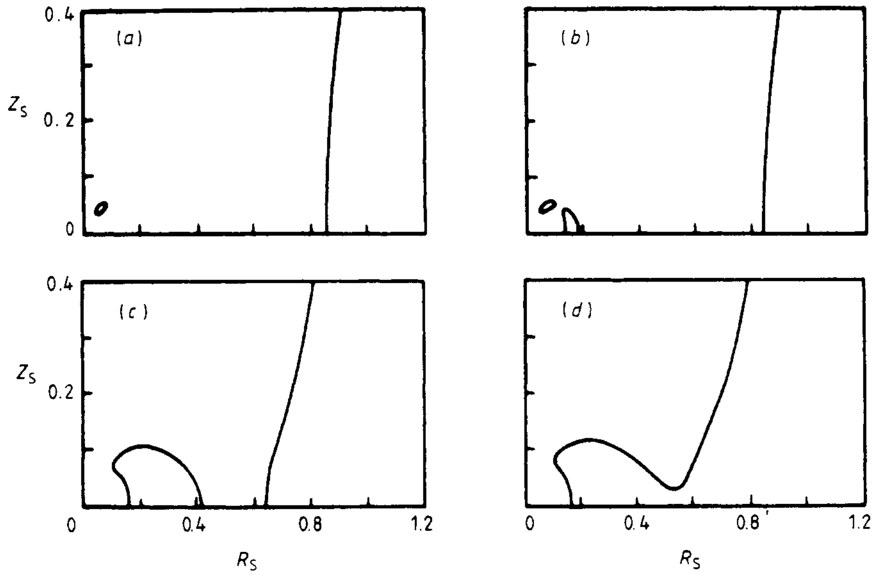


**Figure 2.** 'Allowed' and 'forbidden' regions for charged particles in the combined field of a magnetic dipole and an electric quadrupole, for $\varepsilon = 2$, $\kappa = 0.01$ and increasing $\lambda = (a)$ 0.1, $(b)$ 0.11, $(c)$ 0.2, $(d)$ 0.215.

into three tori, the larger of which cuts through the equatorial plane. In $(c)$ after further increase of the parameter $\lambda$ these three tori have coalesced into one torus. Finally, in $(d)$ this torus has merged with the exterior 'allowed' region and in this case there is no particle storage.

The transition from $(b)$ to $(c)$ of figure 2 corresponds to the development illustrated in figure 3. Parameter values in this sequence are $\varepsilon = 3$ and $\kappa = 0.01$ with $\lambda$ increasing from 0.0913 to 0.0915. This illustrates in great detail the coalescence of the three tori into one torus in a process which may be understood as an increase of the magnetic dipole moment while all the other parameters are kept constant.

Another type of development of the 'allowed' and 'forbidden' regions in the field configuration considered here is shown in figure 4 which has been computed using parameter values $\varepsilon = 10.0$ and $\kappa = -0.1$ with $\lambda$ increasing from $-0.5$ to 0.05. In this sequence two tori are born $(b)$ inside the 'forbidden' region which later merge $(c, d)$ with the 'allowed' exterior region on the right side of the four diagrams.

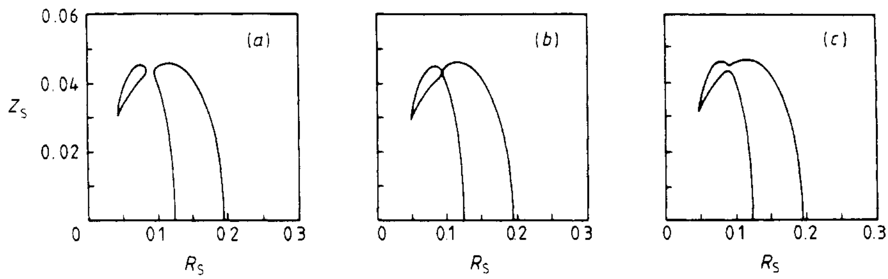


**Figure 3.** Development of three trapping tori into one torus, for $\varepsilon = 3$, $\kappa = 0.01$ and increasing $\lambda = (a)$ 0.0913, $(b)$ 0.0914, $(c)$ 0.0915.

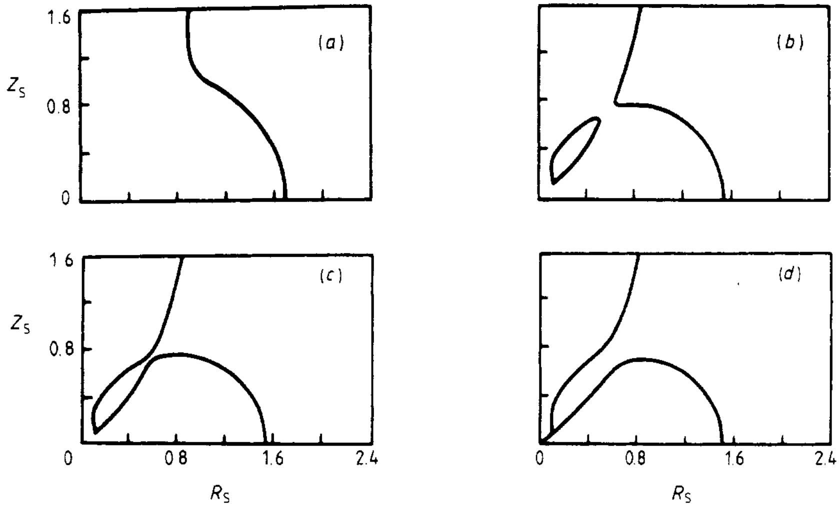


**Figure 4.** Merging of two trapping tori with the external 'allowed' region, for $\varepsilon = 10$, $\kappa = -0.1$ and increasing $\lambda = (a)\ -0.5$, $(b)\ -0.05$, $(c)\ -0.03$, $(d)\ 0.05$.

The aforementioned examples clearly show that within the electromagnetic field configuration considered here one, two or even three disconnected torus-shaped trapping regions may exist, associated with certain ranges of parameter values.

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